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Orthogonal contrasts for rank one common structures

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SUMMARY

The relations between studies in a series, when STATIS (Structuration des Tableaux à Trois Indices de la Statistique) or Dual STATIS is applied, are expressed by the Hilbert-Schmidt products of the operators associated to those studies. Studies in a series are said to have a common structure whenever the matrix of Hilbert-Schmidt products is the sum of a rank one matrix with an error matrix. When such a common structure exists, the components of the structure vector will suffice to locate the studies in the series. Assuming that for all level combinations of L factors there are matched series of studies presenting common structures, standardized orthogonal matrices are used to analyze the influence of those factors on the corresponding structure vectors.

KEY WORDS: STATIS, Dual STATIS, Hilbert-Schmidt products, structure vectors, F tests, orthogonal contrasts, standardized orthogonal matrices.

1. Introduction

Oliveira and Mexia (1998) showed how, given a series of observations $\times$ variables matrices with a common structure, to express that structure by a vector whose components correspond to the matrices in the series. This structure vector condensates the information and enables us to study the global evolution in the series of studies. Moreover, when there where matched series of matrices with common structures, we derived F tests for hypothesis on the corresponding structure vectors and showed how to carry out multiple comparisons. We now extend our analysis to the case when the matched series of matrices correspond to the level combinations of L factors. The hypothesis to be tested concerns the action of those factors on the structure vectors. In deriving these tests we will use orthogonal contrasts. An application of the techniques presented will also be given. This application completes the one given by Oliveira and Mexia (1999b).

2. Statistical techniques

Let the factors have $J_1, \cdots, J_L$ levels. The level combinations can be represented by vectors $\mathbf{j}$ with components $j_l = 1, \cdots, J_l$, $l = 1, \cdots, L$. For each of these $\mathbf{j}$ we will have a series of studies and a matrix $\mathbf{S}_\mathrm{j}$ of Hilbert-Schmidt products between the operators associated with these studies. The estimated structure vector of $\mathbf{S}_\mathrm{j}$ is (Oliveira and Mexia, 1999a) the product $\tilde{\boldsymbol{\beta}}_\mathrm{j}$ of the first eigenvalue λ_j by the first eigenvector $\boldsymbol{\alpha}_\mathrm{j}$ and the sum D_j of squares for error is the sum of squares of the elements of matrix $\mathbf{S}_\mathrm{j} - \lambda_\mathrm{j}\boldsymbol{\alpha}_\mathrm{j}\boldsymbol{\alpha}_j^t$. Oliveira and Mexia (1999) showed that the $\tilde{\boldsymbol{\beta}}_\mathrm{j}$ can be assumed to be normal with mean vectors $\tilde{\boldsymbol{\beta}}_\mathrm{j}$ and variance-covariance matrix $\boldsymbol{\sigma}_\mathrm{j}^2\mathbf{I}_k$ independent from D_j which will be the product of $\boldsymbol{\sigma}_\mathrm{j}^2\mathbf{I}_k$ by a central chi-square with $k(k-1)$ degrees of freedom. If homocedasticity is not rejected we may order the level combinations according to the indexes

$$i(\{\mathbf{j}\}) = j_1 + \sum_{l=2}^{L}(j_l - 1)\prod_{h=1}^{l-1} J_h \tag{1}$$

in order to carry out our analysis. This analysis rests on the use of standardized orthogonal matrices. An $m \times m$ orthogonal matrix is standardized if the elements in its first line are equal to $m^{-1/2}$. Let $\mathbf{P}_1, \cdots, \mathbf{P}_L$ be standardized orthogonal matrices of orders $J_1, \cdots, J_L$. With $\otimes$ denoting the Kronecker matrix product, we now obtain (Mexia, 1988)

$$\overline{\mathbf{P}} = \mathbf{P}_1 \otimes \cdots \otimes \mathbf{P}_L, \tag{2}$$

a standardized orthogonal matrix of order $\bar{J}$. We recall that with $\mathbf{A} = [a_{i,j}]$ of type $r \times s$, we have

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{1,1}B & \cdots & a_{1,s}B \\ \vdots & \ddots & \vdots \\ a_{r,1}B & \cdots & a_{r,s}B \end{bmatrix}. \tag{3}$$

Each line of $\overline{\mathbf{P}}$ is obtained through Kronecker products of lines of the factor matrices $\mathbf{P}_1, ..., \mathbf{P}_L$. We can associate each line of $\bar{\mathbf{P}}$ to the set $\mathbf{C}$ of indexes of the factor matrices that did not contribute with their first line to obtain that line of $\bar{\mathbf{P}}$. Thus matrix $\bar{\mathbf{P}}$ is decomposed in sub-matrices $\mathbf{A}(\mathbf{C})$ constituted by the lines associated to sets $\mathbf{C}$. When $\boldsymbol{\mu}$ is the vector of treatment mean values for a design with L factors, the hypothesis of absence of effects for the j-th factor may be (Mexia,

1988) written as

$$H_0(\{j\}) : \mathbf{A(j)}\boldsymbol{\mu} = 0 \tag{4}$$

and the hypothesis of absence of interactions between the factors with indexes in $\mathbf{C}$ as

$$H_0(\mathbf{C}) : \mathbf{A(j)}\boldsymbol{\mu} = 0. \tag{5}$$

If the design was balanced with r repetitions and g degrees of freedom for the error, the corresponding F tests would have statistics

$$\Im(\mathbf{C}) = \frac{rg}{g(C)} \frac{\|\mathbf{A(C)}\,\mathbf{Y}\,\|^2}{SSE}, \tag{6}$$

with $g(C) = \prod_{l \in C}(J_l - 1)$, $\mathbf{Y}$ the vector of treatment means and SSE the sum of squares for error. We now replace $\boldsymbol{\mu}$ by a vector $\boldsymbol{\varphi}$ whose components correspond to the level combinations. These may be homologue components of the structure vectors. Besides this, $\mathbf{Y}$ will be replaced by the vector of estimators of the components of $\boldsymbol{\varphi}$, SSE by the sum of the sums D_j and r by a convenient standardization coefficient. The method is highly flexible since there is a large range of possibilities in the choice of $\boldsymbol{\varphi}$. This choice must be done taking into account the problem under study. We will show how this may be done in the following application.

3. An application

Oliveira and Mexia (1999b) considered the results of the local elections from 1976 to 1983 in three important districts: Porto, Coimbra and Faro. These districts are situated near the sea: the first in the north, the second in the center and the last one in the south. In that study we tried to analyze the influence of the North-South gradient on the evolution of local political life. Besides the North-South gradient, it is also worthwhile to analyze the effects of a district being on the seaside or in the interior. Thus, a second series of districts is also considered: Bragança in the North, Castelo Branco in the center and Évora in the South. We thus have two factors, one with three and the other with two factors. For each district we had the county results. The counties were the objects, the variables being the results obtained by the political parties: Social Democratic Party (SDP), Socialist Party (SP), Democratic Social Center Party (DSCP), Portuguese Communist Party (PCP), Other parties (OP) (the votes on small parties were pooled together since they have no political relevance), Null Votes (NV) and the abstentions (ABS). The map shown in Figure 1 indicates the location of the six districts.

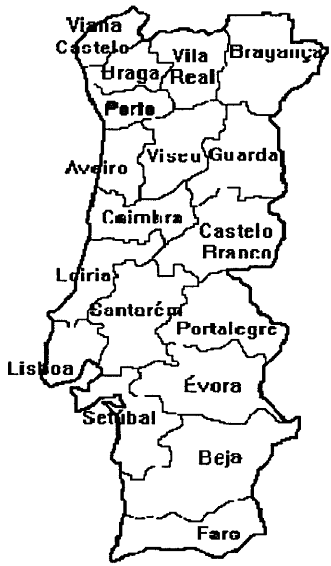

Figure 1. Mainland Portugal

In Table 1 we present the estimated structure vectors for the six series and the corresponding D_j.

Table 1. Estimated structure vectors and sum of errors for the six series

Porto	22.43	21.74	22.13	22.16	21.91	20.92
Coimbra	30.86	28.80	27.25	25.20	24.01	23.74
Faro	24.98	24.22	23.69	25.55	25.43	25.80
Bragança	23.89	24.99	23.38	28.28	25.22	22.52
C.Branco	26.78	26.79	24.52	25.35	24.65	23.75
Evora	23.19	21.97	22.40	22.48	21.92	23.28
D_j	141.50	152.70	97.90	179.20	126.40	52.15

The first, second,..., components of the estimated structure vectors correspond to the first, second,..., elections. For each election we must consider the effects of both factors and their interaction. Moreover, the first factor has three levels: North, Center and South, which can be indexed as 1, 2 and 3. The second factor has only two levels: 1 for Coastal (near the sea), 2 for Interior (far from the sea). If $Y_{i,j}$ is the value of the component under study for the pair (i, j), $i = 1, 2, 3$, $j = 1, 2$, of levels, we can estimate linear and quadratic effects for the first factor, through the expressions

$$\begin{cases} E_L = \frac{1}{2}[(Y_{3,1}+Y_{3,2})-(Y_{1,1}+Y_{1,2})] \\ E_Q = \frac{1}{\sqrt{12}}[(Y_{3,1}+Y_{3,2})-2(Y_{2,1}+Y_{2,2})+(Y_{1,1}+Y_{1,2})] \end{cases}, \tag{7}$$

while the effect of the second factor will be estimated by

$$E_2 = \frac{1}{\sqrt{6}}[(Y_{1,2}+Y_{2,2}+Y_{3,2})-(Y_{1,1}+Y_{2,1}+Y_{3,1})]. \tag{8}$$

We can also estimate the interactions of E_L and E_Q with E_2 through

$$E_L * E_2 = \frac{1}{2}[(Y_{3,2}+Y_{1,2})-(Y_{3,1}+Y_{1,1})], \tag{9}$$

$$E_Q * E_2 = \frac{1}{\sqrt{12}}[(Y_{3,2}-2Y_{2,2}+Y_{1,2})-(Y_{3,1}-2Y_{2,1}+Y_{1,1})]. \tag{10}$$

We now point out that the coefficients in these expressions are the elements of the last five rows of the orthogonal matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{-1}{2} & 0 & \frac{1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{-2}{\sqrt{12}} & \frac{1}{\sqrt{12}} \\ \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & 0 & \frac{-1}{2} & \frac{-1}{2} & 0 & \frac{1}{2} \\ \frac{-1}{\sqrt{12}} & \frac{2}{\sqrt{12}} & \frac{-1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{2}{\sqrt{12}} & \frac{1}{\sqrt{12}} \end{bmatrix}, \tag{11}$$

so that, if $\mathbf{Y}$ has the components $Y_{1,1}$, $Y_{2,1}$, $Y_{3,1}$, $Y_{1,2}$, $Y_{2,2}$, $Y_{3,2}$, our estimates will be the last five components of $\mathbf{PY}$. If $\mathbf{Y}$ is assumed to be normal with variance-covariance matrix $\sigma^2\mathbf{I}_6$ and independent from $D \sim \sigma^2\chi^2_g$ we will have also $\mathbf{PY}$ normal, with variance-covariance matrix $\sigma^2\mathbf{I}_6$ independent from $\mathbf{D}$. It is now straightforward to derive the F tests statistics

$$\begin{cases} \Im_{E_L} = g\dfrac{E_L^2}{D}, \\ \Im_{E_Q} = g\dfrac{E_Q^2}{D}, \\ \Im_{E_2} = g\dfrac{E_2^2}{D}, \\ \Im_{E_L \times E_2} = g\dfrac{(E_L * E_2)^2}{D}, \\ \Im_{E_Q \times E_2} = g\dfrac{(E_Q * E_2)^2}{D}, \end{cases} \tag{12}$$

since in this case $r = 1$. These results could also have been obtained using the general technique described above. Then we would have $\mathbf{P} = \mathbf{P_2} \otimes \mathbf{P_1}$ with

$$\mathbf{P}_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}; \mathbf{P}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}. \tag{13}$$

We then would also obtain global F tests, $\Im_{E_1}$, for the first factor and $\Im_{E_1 * E_2}$ for the interaction between both factors, it being easy to see that

$$\begin{cases} \Im_{E_1} = \frac{1}{2}(\Im_{E_L} + \Im_{E_Q}), \\ \Im_{E_1 * E_2} = \frac{1}{2}(\Im_{E_L \times E_2} + \Im_{E_Q \times E_2}). \end{cases} \tag{14}$$

Actually, we did also perform these tests. Our results are summarized in Tables 2, 3 and 4, where ** and *** denote significance at the 0.01 and 0.001 level, respectively.

Thus there is a highly significant quadratic effect of the first factor in the first three elections (1976, 1979, 1982) which diminishes in the last ones. This is interesting since it shows that the Center had, at first, a different behaviour from the rest of the country.

Table 2. Level totals

North	46.31	46.73	45.51	50.45	47.12	43.49
Center	57.63	55.59	51.77	50.55	48.66	47.50
South	43.01	42.06	42.78	44.21	41.57	41.50
Coastal	72.26	78.09	67.56	66.28	86.90	60.29
Interior	76.02	73.74	67.68	64.99	72.95	59.59

Table 3. Estimated effects and interactions

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
E_L	−1.65	−2.34	−1.37	−3.12	−2.78	−0.97
E_Q	−7.49	−6.47	−4.41	−1.86	−2.49	−2.90
E_2	−0.31	−1.28	−0.22	−2.87	−2.54	−2.72
$E_L * E_2$	−0.45	0.69	−0.38	2.68	0.52	−1.73
$E_Q * E_2$	−3.75	−2.64	−2.52	1.90	1.24	−1.91

Table 4. F tests

	1^{st}	2^{nd}	3^{rd}	4^{th}	5^{th}	6^{th}
E_L	0.65	1.31	0.45	2.34	1.85	0.23
E_Q	13.46***	10.04***	4.66**	0.83	1.49	2.02
E_1	7.05**	5.67**	2.55	1.58	1.67	1.12
E_2	0.02	0.39	0.01	1.97	1.55	1.78
$E_L * E_2$	0.21	0.11	0.04	1.72	0.06	0.72
$E_Q * E_2$	3.38	1.67	1.53	0.86	0.37	0.88
$E_1 * E_2$	1.80	0.89	0.78	1.29	0.22	0.80

Anyone knowledgeable with Portuguese politics knows that there are two dominating parties, PS and PSD, both at the center of the political spectrum. Their predominance has grown progressively so that we are now going to interpret our findings according to this tendency. In the next table we present the pooled percentages of expressed votes in both parties for the six elections.

Table 5. Pooled percentages for the PSD and PS for the six series

	Porto	Coimbra	Faro	Bragança	C.Branco	Evora
1^{st}	88.65	68.39	67.15	54.32	60.01	15.93
2^{nd}	62.98	69.87	65.79	64.71	59.32	39.56
3^{rd}	65.49	72.37	65.94	57.90	60.01	39.03
4^{th}	69.20	74.80	69.19	71.21	68.50	37.53
5^{th}	76.04	81.16	77.43	74.55	77.30	48.45
6^{th}	74.98	83.34	75.26	86.95	82.94	43.01

These percentages are represented in the graphs shown in Figure 2.

Thus, if we exclude the initial drop in Porto there is a steady increase in the pooled percentage for the center parties. Only Évora lags behind in this increase but it follows it. This increase leads to a political dominance of the center throughout the country. Actually, Oliveira and Mexia (1999b) had already referred to this homogenization.

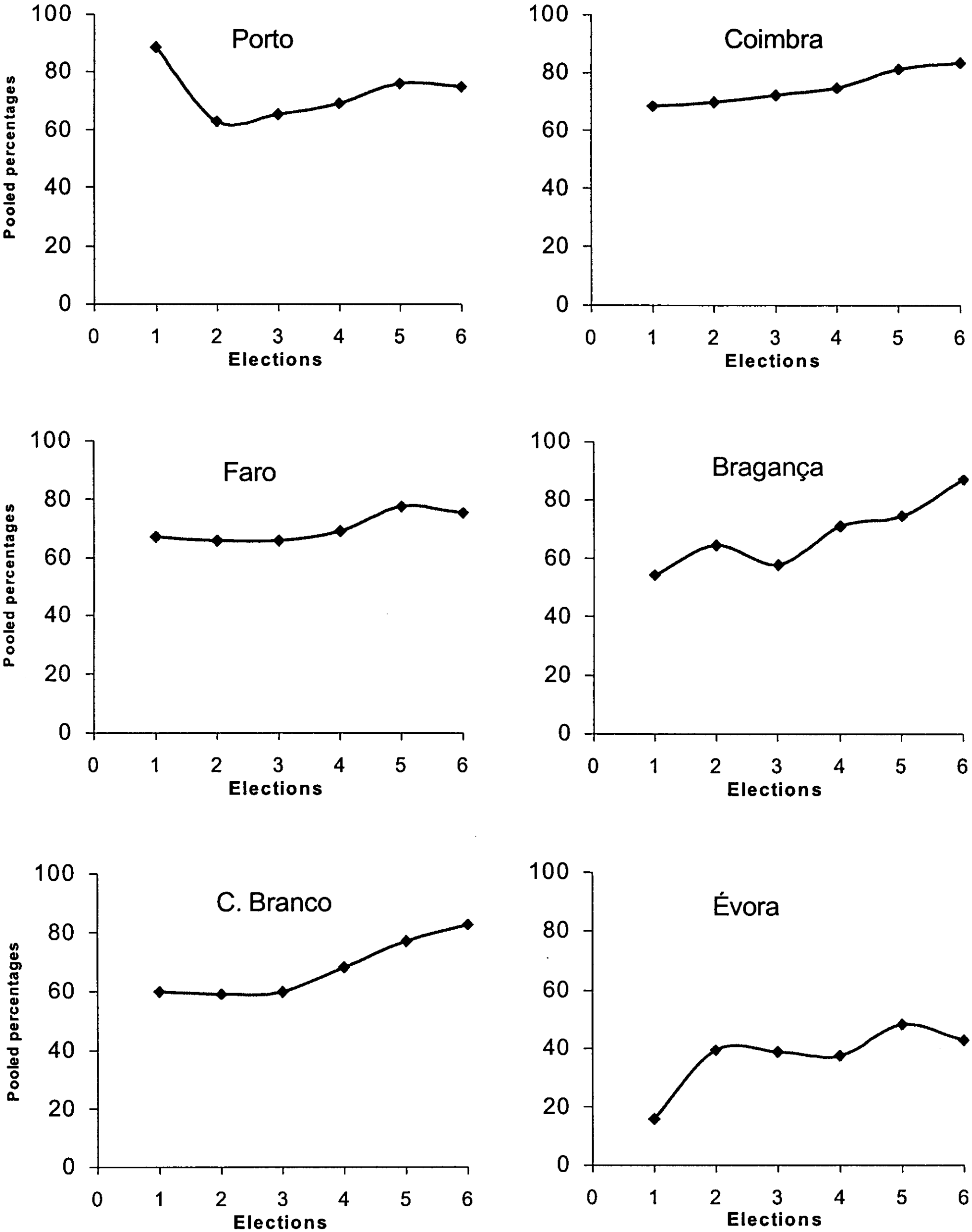

Figure 2. Profiles of pooled percentages for PSD and PS at six elections, for six districts

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Kontrasty ortogonalne dla struktur podobnych rzędu jeden

STRESZCZENIE

W pracy wykorzystano iloczyn Hilberta-Schmidta do opisu relacji pomiędzy wynikami badań stanowiącymi serię, w kontekście zastosowania metody STATIS. Pokazano, że jeżeli badania tworzące serię mają podobną strukturę, to znaczy jeżeli macierz iloczynów Hilberta-Schmidta jest sumą macierzy rzędu jeden oraz macierzy błędów, to składowe wektora struktury wystarczają do identyfikacji poszczególnych badań. Przy założeniu, że każdej kombinacji poziomów czynników doświadczalnych odpowiada badanie, wykorzystano standaryzowane macierze ortogonalne do przeanalizowania wpływu czynników na wektory struktur.

SŁOWA KLUCZOWE: STATIS, iloczyn Hilberta-Schmidta, wektory struktury, test F, kontrasty ortogonalne.